Optimality of a Linear Decision Rule in Discrete Time AK Model*

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Abstract

Surprisingly, formal proof on the optimality of a linear decision rule in the discrete time AK model with a CRRA utility function has not been established in the growth literature while that in the continuous time counterpart is well-established. This note fills such a gap: I provide a formal proof that consumption being linearly related to investment is a sufficient and necessary condition for Pareto optimality in the discrete time AK model.

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1 Introduction

AK model is one of the widely used class of endogenous growth models.[1] Since most growth theories analyze properties of the models in the continuous time framework, properties including optimality are well-established in such a class of models. In particular, continuous time AK model with a CRRA utility function admits a closed form solution because the model's optimality condition yields a first-order nonautonomous linear differential equation in the state variable (Acemoglu (2009)). Since the closed form solution can be easily used to verify whether the transversality condition (hereafter TVC) holds or not, optimality of the linear decision rule in the continuous time AK model has been well-established in the literature.

In contrast, the discrete time AK model with the CRRA utility function yields a difference equation at the optimum, which makes it nontrivial to examine the optimality. While it is not difficult to obtain the equilibrium property that consumption and investment are linearly related with each other as in the continuous counterpart, whether we can rule out other solutions, possibly yielding non-linear relationship between consumption and investment is not a simple question to address. In the sense that AK model is also used in the business cycles literature that usually adopts the discrete time framework (see Barlevy (2004) as an example), it seems to be important to understand the property of the discrete time AK model well for future researches to be based on a concrete theoretical foundation. In doing so, this note provides a formal proof that a unique linear decision rule is sufficient and necessary for Pareto optimality.

In this note, I show that the characterization of the optimal path is consistent with the continuous time AK model: Consumption and investment should be a linear function of each other, hence both variables exhibit linear decision rules with respect to the state variable, K_t, in order to be a sufficient and necessary condition to achieve Pareto optimality. I found that proof for the sufficiency of the linear decision rule for the optimality resembles its counterpart in the continuous time model and proving the necessity (an optimal rule should be linear) involves a step to construct a sequence to ensure that the TVC holds and it is shown that only a linear decision rule can satisfy this property.

In the sense that I study the property of the endogenous growth model in the discrete time framework, this paper is in line with Le Van, Morhaim and Dimaria (2002) that presents a discrete time version of

[1] While Jones (1995) criticized that predictions of the AK model is not consistent with the empirical evidence, McGrattan (1998) argued that the AK model may be a good approximation of the real economy if time span for the data is extended.

the Romer 1986 model. Hence, this paper contributes to the literature on economic growth by providing a theoretical groundwork for future works utilizing the AK model. In addition, I expect the steps taken in the proof can be used in the future works adopting dynamic models of the same class in which variables relevant to the TVC permanently grow.

2 Model Preliminaries

The first and second theorems of welfare economics hold in this economy since I consider an environment with no distortions or frictions, implying that I can directly set up a social planner's problem as follows.

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \tag{2.1}$$

subject to

$$C_t + K_{t+1} = AK_t + (1-\delta)K_t \tag{2.2}$$

where $\beta \in (0,1)$ is a discount factor, $\sigma > 0$ is a CRRA parameter, $A > 0$ is a technology parameter, and $\delta \in (0,1)$ is the rate of depreciation for capital. Assume that $K_0 > 0$ is given.

3 Optimality of the Discrete Time AK Model: Sufficient and Necessary Conditions

I first define the optimal allocation as follows.

Definition (Pareto Optimality). *A sequence of allocation, $\{C_t, K_{t+1}\}_{t=0}^{\infty}$, is Pareto optimal if and only if it is the solution to the problem (2.1) subject to the feasibility condition (equation (2.2)).*

The following proposition provides a sufficient and necessary condition for the Pareto optimal allocation.[2]

Proposition 1 (Optimal Allocation). *The following system of equations is a sufficient and necessary condition for the sequence of allocation, $\{C_t^*, K_{t+1}^*\}_{t=0}^{\infty}$, to be Pareto optimal.*

[2] It can be easily verified that equations (3.1) and (3.2) are equivalent to those obtained from the continuous time counterpart if I let Δt (difference between t and $t+1$) converge toward zero.

$$\begin{aligned}
\textit{(Optimality condition)} \quad \frac{C^*_{t+1}}{C^*_t} &= [\beta(A+1-\delta)]^{\frac{1}{\sigma}} && (3.1)\\
\textit{(Feasibility condition)} \quad C^*_t + K^*_{t+1} &= (A+1-\delta)K^*_t && (3.2)\\
\textit{(TVC)} \quad \lim_{t\to\infty} \beta^t K^*_{t+1} (C^*_t)^{-\sigma} &= 0 && (3.3)
\end{aligned}$$

Proof. Refer to any textbooks for economic growth (for instance, see Acemoglu (2009) and/or Barro and Sala-i-Martin (2004)). □

Since I am interested in the economy with positive growth, the following restriction on parameters will be assumed.

Assumption 1 (Non-Negative Growth)**.**

$$\beta(A+1-\delta) > 1 \tag{3.4}$$

Throughout the proof, the following restriction on the parameters will be additionally assumed for the existence of the optimal path[3]:

Assumption 2 (Existence of a Linear Rule)**.**

$$[\beta(A+1-\delta)]^{\frac{1}{\sigma}} < A+1-\delta \tag{3.5}$$

The following proposition is the main result and the proof for the sufficiency part directly follows since provision of the formal proof is the main contribution of this paper. To save space, proof for the necessity part is omitted in the main text and can be found at Appendix A since the proof is in line with that for the sufficient condition.

Proposition 2 (Pareto Optimality with a Linear Rule in the Discrete Time AK Economy)**.** *In the discrete time AK economy, Pareto optimum is achieved if and only if consumption (C_t) and investment (K_{t+1}) are linearly related. In particular, the following should hold:*

$$C_t = \frac{\phi}{1-\phi} K_{t+1} \tag{3.6}$$

[3] If log utility ($\sigma = 1$) is considered, this assumption is redundant.

where $\phi \equiv 1 - \beta^{\frac{1}{\sigma}} (A + 1 - \delta)^{\frac{1}{\sigma}-1}$.

The proof for the sufficient condition resembles that of the continuous time counterpart in the sense that the linear rule that satisfies equations (3.1) and (3.2) is shown to satisfy the TVC. The key to prove the necessity part is constructing a sequence of $\{K_{t+1}C_t^{-\sigma}\}$, which is not necessarily increasing or decreasing, that is in the TVC. For interested readers, I refer to the appendix for the formal proof for the necessary condition.

Proof. Since C_t and K_{t+1} are linearly related as $C_t = \frac{\phi}{1-\phi}K_{t+1}$, I can make the guess that $C_t = \phi(A + 1 - \delta)K_t$ and $K_{t+1} = (1 - \phi)(A + 1 - \delta)K_t$ using the feasibility condition (3.2). The optimality condition (equation (3.1)) implies

$$\underbrace{\frac{C_{t+1}}{C_t}}_{=\frac{K_{t+1}}{K_t}=(1-\phi)(A+1-\delta)} = [\beta(A + 1 - \delta)]^{\frac{1}{\sigma}} \tag{3.7}$$

Then $\phi \equiv 1 - \beta^{\frac{1}{\sigma}} (A + 1 - \delta)^{\frac{1}{\sigma}-1}$ can be easily obtained. Assumption 2 ensures $\phi \in (0, 1)$.

Since the above solution satisfies the optimality condition and feasibility condition, checking if the TVC holds is the remaining part of the proof. The following lemma is useful for the proof:

Lemma 1 (Optimal Consumption as a Function of Initial Capital)**.** *Let* $K_0 > 0$ *be the initial capital level. Then optimal consumption can be described as follows.*

$$C_t = \frac{\phi}{1-\phi}\left[\beta^{\frac{1}{\sigma}}(A + 1 - \delta)^{\frac{1}{\sigma}}\right]^{t+1} K_0 \tag{3.8}$$

Proof. Since $C_t = \phi(A + 1 - \delta)K_t$ and $K_t = (1 - \phi)(A + 1 - \delta)K_{t-1}$, substitution yields $C_t = \phi(1 - \phi)(A + 1 - \delta)^2K_{t-1}$. Then recursive substitution yields $C_t = \phi(1 - \phi)^t(A + 1 - \delta)^{t+1}K_0$ and using the definition of ϕ provides the above solution. □

Now I can show that the TVC holds with the linear rule. For simplicity of the argument, I consider $\beta^t K_{t+1}C_t^{-\sigma}$ at first and then will take it to the limit.

$$
\begin{aligned}
\beta^t K_{t+1} C_t^{-\sigma} &= \beta^t \underbrace{\frac{K_{t+1}}{C_t}}_{=\frac{1-\phi}{\phi}} C_t^{1-\sigma} \\
&= \frac{1-\phi}{\phi}\beta^t \left[\frac{\phi}{1-\phi}\left[\beta^{\frac{1}{\sigma}}(A+1-\delta)^{\frac{1}{\sigma}}\right]^{t+1} K_0\right]^{1-\sigma} \\
&= \frac{1-\phi}{\phi}\left[\underbrace{\beta^{\frac{1}{\sigma}}(A+1-\delta)^{\frac{1-\sigma}{\sigma}}}_{=1-\phi}\right]^t \left[\frac{\phi}{1-\phi}\left[\beta^{\frac{1}{\sigma}}(A+1-\delta)^{\frac{1}{\sigma}}\right] K_0\right]^{1-\sigma} \\
&= \underbrace{\frac{1-\phi}{\phi}\left[\frac{\phi}{1-\phi}\left[\beta^{\frac{1}{\sigma}}(A+1-\delta)^{\frac{1}{\sigma}}\right] K_0\right]^{1-\sigma}}_{<\infty}(1-\phi)^t \\
&\propto (1-\phi)^t
\end{aligned} \tag{3.9}
$$

Hence TVC holds under the Assumption 2, which completes the proof.

$$
\lim_{t\to\infty} \beta^t K_{t+1} C_t^{-\sigma} \propto \lim_{t\to\infty}(1-\phi)^t = 0 \tag{3.10}
$$

□

As a result, the linear decision rule described by the equation (3.6) is a sufficient and necessary condition for the allocation obtained in the AK model to be Pareto optimal.

The corollary on the balanced growth path property directly follows, which is the property that also holds in the continuous time counterpart.

Corollary 1. *Balanced growth path property holds at the optimum.*

Proof. Consumption grows at a constant rate (equation (3.1)). Linear rule (equation (3.6)) implies that capital should grow at the constant rate and hence output. □

4 Concluding Remark

As is noted in Licandro, Puch and Ruiz (2018) and Gómez (2014), equilibrium properties of continuous time and discrete time models do not always coincide. Given that the two models are usually used in different contexts (to explain growth and/or business cycles), it is quite important to understand the properties of both models. Thus this paper contributes to the literature on economic growth in this respect by providing a formal proof on the sufficiency and necessity of the linear decision rule in the discrete time AK framework.

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A Appendix. Proof for Necessary Condition

Suppose that $\{\hat{C}_t, \hat{K}_{t+1}\}$ is the optimal rule that satisfies Proposition 1. Multiplying $\hat{C}_t^{-\sigma}$ to the feasibility condition leads to the following equation after rearranging the terms:

$$\hat{K}_{t+1}\hat{C}_t^{-\sigma} = (A+1-\delta) \underbrace{\hat{K}_t\hat{C}_t^{-\sigma}}_{\equiv \hat{K}_t\hat{C}_{t-1}^{-\sigma}\left(\frac{\hat{C}_t}{\hat{C}_{t-1}}\right)^{-\sigma}} - \hat{C}_t^{1-\sigma} \tag{A.1}$$

From the optimality condition (equation (3.1)), $\frac{\hat{C}_t}{\hat{C}_{t-1}} = [\beta(A+1-\delta)]^{\frac{1}{\sigma}}$. Hence the feasibility condition becomes

$$\hat{K}_{t+1}\hat{C}_t^{-\sigma} = \frac{1}{\beta}\hat{K}_t\hat{C}_{t-1}^{-\sigma} - \hat{C}_t^{1-\sigma} \tag{A.2}$$

Notice that this equation describes the sequence of $\{\hat{K}_{t+1}\hat{C}_t^{-\sigma}\}$.

The next Lemma would be useful for the proof:

Lemma 2 (Optimal Consumption as a Function of Initial Consumption)**.** *Let $\hat{C}_0 > 0$ be the initial consumption level optimally chosen by the planner. Then optimal consumption can be described as follows.*

$$\hat{C}_t = [\beta(A+1-\delta)]^{\frac{t}{\sigma}}\,\hat{C}_0 \tag{A.3}$$

Proof. Recursive substitution of the optimality condition (3.1) yields the above expression. □

Then $\hat{C}_t^{1-\sigma} = \omega^t\hat{C}_0^{1-\sigma}$ with $\omega \equiv \{\beta(A+1-\delta)\}^{\frac{1-\sigma}{\sigma}}$. Substituting this expression into the equation (A.2):

$$\begin{aligned}\hat{K}_{t+1}C_t^{-\sigma} &= \frac{1}{\beta}\underbrace{\hat{K}_t\hat{C}_{t-1}^{-\sigma}}_{=\frac{1}{\beta}\hat{K}_{t-1}\hat{C}_{t-2}^{-\sigma}-\omega^{t-1}\hat{C}_0^{1-\sigma}} - \omega^t\hat{C}_0^{1-\sigma} \\ &= \left(\frac{1}{\beta}\right)^2\hat{K}_{t-1}\hat{C}_{t-2}^{-\sigma} - \hat{C}_0^{1-\sigma}\left(\omega^t + \frac{1}{\beta}\omega^{t-1}\right)\end{aligned} \tag{A.4}$$

One can substitute the expression for $\{\hat{K}_{t+1}\hat{C}_t^{-\sigma}\}$ recursively and obtain the following expression for $\hat{K}_{t+1}C_t^{-\sigma}$.

$$\hat{K}_{t+1}\hat{C}_t^{-\sigma} = \left(\frac{1}{\beta}\right)^t \hat{K}_1\hat{C}_0^{-\sigma} - \hat{C}_0^{1-\sigma}\left(\underbrace{\omega^t + \frac{1}{\beta}\omega^{t-1} + \cdots + \left(\frac{1}{\beta}\right)^{t-1}\omega}_{\equiv\frac{\omega^t\left(1-\left(\frac{1}{\beta\omega}\right)^t\right)}{1-\frac{1}{\beta\omega}}}\right) \tag{A.5}$$

The next step is to verify that $\hat{K}_{t+1}\hat{C}_t^{-\sigma}$, the term in the TVC, converges toward zero if it is multiplied by β^t.

$$\begin{aligned}
\beta^t\hat{K}_{t+1}\hat{C}_t^{-\sigma} &= \beta^t\left[\left(\frac{1}{\beta}\right)^t \hat{K}_1\hat{C}_0^{-\sigma} - \hat{C}_0^{1-\sigma}\frac{\omega^t\left(1-\left(\frac{1}{\beta\omega}\right)^t\right)}{1-\frac{1}{\beta\omega}}\right] \\
&= \hat{K}_1\hat{C}_0^{-\sigma} - \hat{C}_0^{1-\sigma}\frac{\beta\omega}{\beta\omega-1}(\beta\omega)^t\left(1-\left(\frac{1}{\beta\omega}\right)^t\right) \\
&= \hat{K}_1\hat{C}_0^{-\sigma} - \hat{C}_0^{1-\sigma}\frac{\beta\omega}{\beta\omega-1}\left(\underbrace{(\beta\omega)^t - 1}_{=(\beta\omega-1)\left[1+\beta\omega+\cdots+(\beta\omega)^{t-1}\right]}\right) \\
&= \hat{K}_1\hat{C}_0^{-\sigma} - \hat{C}_0^{1-\sigma}\beta\omega\left[\underbrace{1+\beta\omega+\cdots+(\beta\omega)^{t-1}}_{=\frac{\left(1-(\beta\omega)^t\right)}{1-\beta\omega}}\right] \\
&= \hat{K}_1\hat{C}_0^{-\sigma} - \hat{C}_0^{1-\sigma}\beta\omega\frac{\left(1-(\beta\omega)^t\right)}{1-\beta\omega}
\end{aligned} \tag{A.6}$$

Using the definition of ω, $\beta\omega = \beta\left\{\beta(A+1-\delta)\right\}^{\frac{1-\sigma}{\sigma}} = \beta^{\frac{1}{\sigma}}(A+1-\delta)^{\frac{1-\sigma}{\sigma}} \equiv 1-\phi$. Then $\beta\omega\frac{\left(1-(\beta\omega)^t\right)}{1-\beta\omega} = \frac{(1-\phi)}{\phi}\left(1-(1-\phi)^t\right)$.

Thus

$$\begin{aligned}\lim_{t\to\infty}\beta^t \hat{K}_{t+1}\hat{C}_t^{-\sigma} &= \lim_{t\to\infty}\hat{C}_0^{-\sigma}\left[\hat{K}_1 - \frac{(1-\phi)}{\phi}\left(1-\underbrace{(1-\phi)^t}_{\to 0 \text{ when } t\to\infty}\right)\hat{C}_0\right] \\ &= \lim_{t\to\infty}\hat{C}_0^{-\sigma}\left[\hat{K}_1 - \frac{(1-\phi)}{\phi}\hat{C}_0\right] \qquad \text{(A.7)}\end{aligned}$$

Since $\hat{C}_0 \in (0,\infty)$ (if $\hat{C}_0 = 0$, the optimality condition (3.1) yields $\hat{C}_t = 0$ for all t, which is not optimal given that marginal utility of consumption diverges toward infinity when consumption is near zero), the TVC holds only when $\hat{K}_1 = \frac{(1-\phi)}{\phi}\hat{C}_0$, implying that investment and consumption chosen by the planner should be linearly related with each other.

To further show that the above property holds for any t, I will use mathematical induction. Suppose that $\hat{K}_t = \frac{1-\phi}{\phi}\hat{C}_{t-1}$ for some $t \geq 1$. The following lemma is helpful for the proof:

Lemma 3 (Optimal Rule for Capital Growth)**.** *Along the optimal path, the following should hold.*

$$\hat{K}_{t+1} = \beta^{\frac{1}{\sigma}}(A+1-\delta)^{\frac{1}{\sigma}}\hat{K}_t \qquad \text{(A.8)}$$

Proof. From equation (3.1), $\hat{C}_t = [\beta(A+1-\delta)]^{\frac{1}{\sigma}}\hat{C}_{t-1}$. From the feasibility condition (3.2),

$$\begin{aligned}\hat{K}_{t+1} &= (A+1-\delta)\hat{K}_t - \hat{C}_t \\ &= (A+1-\delta)\hat{K}_t - [\beta(A+1-\delta)]^{\frac{1}{\sigma}}\hat{C}_{t-1} \\ &= (A+1-\delta)\hat{K}_t - [\beta(A+1-\delta)]^{\frac{1}{\sigma}}\frac{\phi}{1-\phi}\hat{K}_t \\ &= \left[(A+1-\delta) - [\beta(A+1-\delta)]^{\frac{1}{\sigma}}\frac{1-\beta^{\frac{1}{\sigma}}(A+1-\delta)^{\frac{1}{\sigma}-1}}{\beta^{\frac{1}{\sigma}}(A+1-\delta)^{\frac{1}{\sigma}-1}}\right]\hat{K}_t \\ &= \beta^{\frac{1}{\sigma}}(A+1-\delta)^{\frac{1}{\sigma}}\hat{K}_t \qquad \text{(A.9)}\end{aligned}$$

□

Hence, $\frac{\hat{C}_t}{\hat{C}_{t-1}} = \frac{\hat{K}_{t+1}}{\hat{K}_t} = \beta^{\frac{1}{\sigma}}(A+1-\delta)^{\frac{1}{\sigma}}$, implying $\hat{K}_{t+1} = \frac{1-\phi}{\phi}\hat{C}_t$ for any t. As this relationship holds for $t=0$, $\hat{K}_{t+1} = \frac{1-\phi}{\phi}\hat{C}_t$ for all $t \geq 0$.