101562: Intermediate Macroeconomics
Note on Lagrangian Method

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Instructor: Myungkyu Shim

Consider the usual consumer utility maximization problem:

$$\max_{x_1,x_2} u(x_1,x_2)$$

subject to

$$p_1x_1 + p_2x_2 = Y$$

where $p_i > 0$ is the price of good $i \in \{1,2\}$ and $Y > 0$ is the income of the consumer. We assume that (1) the utility function, u, is (strictly) increasing, (strictly) concave, and at least twice differentiable and (2) consumers are price takers.

Notice that the above model is a 'constrained' maximization problem; the consumer should choose the 'finite' optimal consumption plan subject to the finite income. However, the constrained maximization problem itself is not easy to solve for. So we introduce the 'Lagrangian method', which changes the constrained problem into the 'unconstrained' problem.

This note shows the steps for applying the Lagrangian method, which will be a useful tool in this class.

STEP 1: Form a Lagrangian.

Lagrangian is the sum of (1) the objective function and (2) the terms obtained by multiplying the Lagrangian multipliers with the associated (rearranged) constraints. Then

$$\mathcal{L} = u(x_1,x_2) + \lambda\left[Y - p_1x_1 - p_2x_2\right]$$

for the above problem.

Firstly, we arrange the (budget) constraints so that all the variables are in the same side. Then we multiply the rearranged term with the Lagrangian multiplier (λ)[1][2] as above. Now we have an unconstrained problem whose objective function is $\mathcal{L}$ and the associated solution to the problem is identical to that of the original constrained problem.

STEP 2: Derive the FOCs (First Order Conditions).

With n choice variables (2 choice variables in our problem), we differentiate the Lagrangian with respect to each choice variable:

$$(FOC.1)\ \frac{\partial \mathcal{L}}{\partial x_1} = mu_1\left(x_1,x_2\right) - p_1\lambda = 0$$

[1] $\lambda > 0$ when the budget constraint is rewritten as above.

[2] The Lagrangian multiplier is usually interpreted as the 'shadow price' in Economics.

$$(FOC.2)\ \frac{\partial \mathcal{L}}{\partial x_2} = mu_2\left(x_1, x_2\right) - p_2\lambda = 0$$

where $mu_i\left(x_1, x_2\right) = \frac{\partial u(x_1,x_2)}{\partial x_i}$, which is the marginal utility from consuming good i. Notice that the objective function is concave so that the solution that satisfies $(FOC.1)$ and $(FOC.2)$ is the maximizer of the problem. Notice further that we have the same λ in both equations. We can then combine the two equations to obtain

$$\frac{mu_1\left(x_1, x_2\right)}{mu_2\left(x_1, x_2\right)} = \frac{p_1}{p_2} \tag{1}$$

The left hand side of the above equation is the marginal rate of substitution between the two goods $(= MRS_{1,2})$. As we learned in the intermediate micro class, the above equation implies that the optimal consumption plan is obtained when the consumer's subjective valuation of the two goods $(MRS_{1,2})$ is equal to the objective valuation of the two goods at the market $(\frac{p_1}{p_2})$.

STEP 3: Obtain the Solution.

Recall that we have two choice variables (= two unknowns) but there is only one equilibrium condition, equation (1). Note that the solution should also satisfy the budget constraint. Hence, the optimal consumption plan is required to satisfy the following two equations at the same time:

$$\frac{mu_1\left(x_1, x_2\right)}{mu_2\left(x_1, x_2\right)} = \frac{p_1}{p_2} \tag{2}$$

$$p_1 x_1 + p_2 x_2 = Y \tag{3}$$

Now we have two equations and two unknowns, which provides us the exact solution of the problem (if the solution exists[3]).

EXAMPLE

For instance, if $u(x_1, x_2) = \alpha \ln x_1 + \beta \ln x_2$, the equation (2) becomes

$$\frac{\alpha x_2}{\beta x_1} = \frac{p_1}{p_2} \ \Leftrightarrow p_2 x_2 = \frac{\beta}{\alpha} p_1 x_1$$

Substituting this equation into the equation (3) yields

$$x_1^* = \frac{\alpha}{\alpha+\beta}\frac{Y}{p_1} \text{ and } x_2^* = \frac{\beta}{\alpha+\beta}\frac{Y}{p_1}$$

where x_1^* and x_2^* are the solutions of the utility maximization problem.

[3] We need further assumptions on the utility function for the existence of the solutions, which are satisfied with the majority of the utility functions that we will use in the class.